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Drinfeld comultiplication and vertex operators

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Abstract

For the current realization of the quantum affine algebras, Drinfeld gave a simple comultiplication of the quantum current operators. With this comultiplication, we study the related vertex operators for the case of $U_q(\hat{\mathfrak{sl}}_n)$ and give an explicit bosonization of these new vertex operators. We use these vertex operators to construct the quantum current operators of $U_q(\hat{\mathfrak{sl}}_n)$ and discuss its connection with quantum boson–fermion correspondence.

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1. Introduction

Discovered by Drinfeld [Dr1] and Jimbo [J1], Quantum group as a Hopf algebra presents a new structure in both mathematics and physics. One of the fundamental features of this structure comes from its comultiplication, which is basically related to all the new concepts in this structure, such as R-matrix, etc. The most widely used comultiplication is given with the definition of a quantum group by the basic generators and the relations based on the data coming from the corresponding Cartan matrix. However, in the case of quantum affine algebras, there is a neglected aspect of the story. In this case, Drinfeld presented a different formulation of quantum affine algebras with generators in the form of current operators [Dr3], for which he proposed another comultiplication formula [DF1] based on the current formulation. The fundamental feature of this comultiplication is its simplicity, as opposed to the comultiplication formula induced from the conventional comultiplication

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which simply cannot be written in a closed form with these current operators. However, this new comultiplication is seldom used beyond its definition. We propose to use this comultiplication formula to study the vertex operators coming from such a comultiplication.

In this paper, we will first study the vertex operators for the fundamental representations at the level 1 for the case of $U_q(\hat{\mathfrak{sl}}_n)$. Using the Frenkel–Jing construction of level 1 representations of $U_q(\hat{\mathfrak{sl}}_n)$ and the conventional comultiplication, the Kyoto group [DFJMN, DO, Ko] studied the related vertex operators and its application to XXZ model in statistical mechanics. They obtained the bosonization of the vertex operators, which is partially incomplete because the bosonization can be done only for one component of the vertex operators while the rest are implicit. With the new comultiplication, we give a complete and simple formula for the related intertwiners. In the classical case, these vertex operators are the corresponding Clifford algebras, from which we can reconstruct the level 1 representations, otherwise called spinor representations. This gives the boson–fermion correspondence [F1, Dil]. With the new formula of the vertex operators, which can be interpreted as the deformed fermions, we recover the current operators of $U_q(\hat{\mathfrak{sl}}_n)$.

The main paper contains two sections. In Section 2, we will present the basic definitions and prove the new comultiplication formulae. In Section 3, we will present the Frenkel–Jing [FJ] construction and present the explicit formulas for all the corresponding vertex operators. We will also give some formulae of the construction of the quantum current from these vertex operators and discuss its connection with quantum boson–fermion correspondence. In this paper, we will restrict ourselves to the case of $U_q(\hat{\mathfrak{sl}}_n)$. The extension of these results to other cases is straightforward and will be the subject in a subsequent paper.

2. Drinfeld comultiplication

The first definition of quantum groups as a Hopf algebra given by Drinfeld [Dr1] and Jimbo [J1] is given in terms of basic generators and relations with corresponding Cartan matrices. For the case of quantum affine algebras, Drinfeld gave a realization in terms of operators in the form of current [Dr3]. We will first present this realization for the case of $U_q(\hat{\mathfrak{sl}}_n)$.

Let $A = (a_{ij})$ be the Cartan matrix of type A_{n-1}.

Definition 2.1. The algebra $U_q(\hat{\mathfrak{sl}}_n)$ is an associative algebra with unit 1 and the generators: $\varphi_i(-m)$, $\psi_i(m)$, $x_i^{\pm}(l)$, for $i = i, \ldots, n-1$, $l \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$ and a central element c. Let z be a formal variable and $x_i^{\pm}(z) = \sum_{l\in\mathbb{Z}} x_i^{\pm}(l)z^{-l}$, $\varphi_i(z) = \sum_{m\in-\mathbb{Z}_{\geq 0}} \varphi_i(m)z^{-m}$ and $\psi_i(z) = \sum_{m\in\mathbb{Z}_{\geq 0}} \psi_i(m)z^{-m}$. In terms of the formal variables, the defining relations are:

$$\varphi_i(z)\varphi_j(w) = \varphi_j(w)\varphi_i(z),$$

$$\psi_i(z)\psi_j(w) = \psi_j(w)\psi_j(z),$$

$$\varphi_i(z)\psi_j(w)\varphi_i(z)^{-1}\psi_j(w)^{-1} = \frac{g_{ij}(z/wq^{-c})}{g_{ij}(z/wq^{c})},$$

$$\varphi_i(z)x_j^{\pm}(w)\varphi_i(z)^{-1} = g_{ij}(z/wq^{\mp c/2})^{\pm 1}x_j^{\pm}(w),$$

$$\psi_i(z)x_j^{\pm}(w)\psi_i(z)^{-1} = g_{ij}(w/zq^{\mp c/2})^{\mp 1}x_j^{\pm}(w),$$

$$[x_i^+(z), x_j^-(w)] = \frac{\delta_{i,j}}{q-q^{-1}}\{\delta(z/wq^{-c})\psi_i(wq^{c/2}) - \delta(z/wq^{c})\varphi_i(zq^{c/2})\},$$

$$(z - q^{\pm a_{ij}}w)x_i^{\pm}(z)x_j^{\pm}(w) = (q^{\pm a_{ij}}z - w)x_j^{\pm}(w)x_i^{\pm}(z),$$

$$[x_i^{\pm}(z), x_j^{\pm}(w)] = 0 \quad \text{for } a_{ij} = 0,$$

$$x_i^{\pm}(z_1)x_i^{\pm}(z_2)x_j^{\pm}(w) - (q+q^{-1})x_i^{\pm}(z_1)x_j^{\pm}(w)x_i^{\pm}(z_2) + x_j^{\pm}(w)x_i^{\pm}(z_1)x_i^{\pm}(z_2)$$
$$+\{z_1 \leftrightarrow z_2\} = 0 \quad \text{for } a_{ij} = -1,$$

where

$$\delta(z) = \sum_{k\in\mathbb{Z}} z^k, \qquad g_{ij}(z) = \frac{q^{a_{ij}}z - 1}{z - q^{a_{ij}}} \quad \text{about } z = 0.$$

In [Dr3], Drinfeld only gave the formulation of the algebra. If we extend the conventional comultiplication to these current operators, the result would be a very complicated formula which cannot be written in a closed form with only these current operators. However, Drinfeld also gave the Hopf algebra structure for such a formulation in an unpublished note [DF1].

Theorem 2.2. *The algebra $U_q(\hat{\mathfrak{sl}}_n)$ has a Hopf algebra structure, which is given by the following formulae.*

Coproduct Δ:

(0) $\Delta(q^{c/2}) = q^{c/2} \otimes q^{c/2}$,

(1) $\Delta(x_i^+(z)) = x_i^+(z) \otimes 1 + \varphi_i(zq^{c_1/2}) \otimes x_i^+(zq^{c_1})$,

(2) $\Delta(x_i^-(z)) = 1 \otimes x_i^-(z) + x_i^-(zq^{c_2}) \otimes \psi_i(zq^{c_2/2})$,

(3) $\Delta(\varphi_i(z)) = \varphi_i(zq^{-c_2/2}) \otimes \varphi_i(zq^{c_1/2})$,

(4) $\Delta(\psi_i(z)) = \psi_i(zq^{c_2/2}) \otimes \psi_i(zq^{-c_1/2})$,

where $q^{c_1/2} = q^{c/2} \otimes 1$ and $q^{c_2/2} = 1 \otimes q^{c/2}$.

Counit ε:

$$\epsilon(q^c) = 1, \qquad \varepsilon(\varphi_i(z)) = \varepsilon(\psi_i(z)) = 1, \qquad \varepsilon(x_i^{\pm}(z)) = 0.$$

Antipode a:

(0) $a(q^c) = q^{-c}$,

(1) $a(x_i^+(z)) = -\varphi_i(zq^{-c/2})^{-1}x_i^+(zq^{-c})$,

(2) $a(x_i^-(z)) = -x_i^-(zq^{-c})\psi_i(zq^{-c/2})^{-1}$,

(3) $a(\varphi_i(z)) = \varphi_i(z)^{-1}$,

(4) $a(\psi_i(z)) = \psi_i(z)^{-1}$.

It is therefore clear that the comultiplication structure requires certain completion on the tensor space. For certain representations, such as the n-dimensional representations of $U_q(\hat{\mathfrak{sl}}_n)$, which will be presented in Section 3, this comultiplication may not be well defined. Nevertheless, in the case of any two highest weight representations, this comultiplication is well defined and is already used in [DM] to study the poles and zeros of the current operators for integrable representations. We will further present the proof for Theorem 2.2 for the case of $U_q(\hat{\mathfrak{sl}}_2)$. However, this proof should be completely attributed to Drinfeld [DF1].

Proof of Theorem 2.2 (for the case of $U_q(\hat{\mathfrak{sl}}_2)$). For the comultiplication above we have that

(1) $\Delta(x_1^+(z)) = x_1^+(z)\otimes 1 + \varphi_1(zq^{c_1/2})\otimes x_1^+(zq^{c_1})$,

(2) $\Delta(x_1^-(z)) = 1\otimes x_1^-(z) + x_1^-(zq^{c_2})\otimes \psi_1(zq^{c_2/2})$,

(3) $\Delta(\varphi_1(z)) = \varphi_1(zq^{-c_2/2})\otimes \varphi_1(zq^{c_1/2})$,

(4) $\Delta(\psi_1(z)) = \psi_1(zq^{c_2/2})\otimes \psi_1(zq^{-c_1/2})$.

It is clear that

$$\Delta\varphi_1(z)\Delta\varphi_1(w) = \Delta\varphi_1(w)\Delta\varphi_1((z), \qquad \Delta\psi_1(z)\Delta\psi_1(w) = \Delta\psi_1(w)\Delta\psi_1(z).$$

Then

$$\begin{aligned}
&\Delta\varphi_1(z)\Delta\psi_1(w)\Delta\varphi_1(z)^{-1}\Delta\psi_1(w)^{-1}\\
&\quad= \varphi_1(zq^{-c_2/2})\psi_1(wq^{c_2/2})\varphi_1(zq^{-c_2/2})^{-1}\psi_1(wq^{c_2/2})^{-1}\\
&\qquad\otimes\ \varphi_1(zq^{c_1/2})\psi_1(wq^{-c_1/2})\varphi_1(zq^{c_1/2})^{-1}\psi_1(wq^{-c_1/2})^{-1}\\
&\quad= \frac{g_{11}(z/wq^{-c_1}q^{-c_2})}{g_{11}(z/wq^{c_1}q^{-c_2})}\,\frac{g_{11}(z/wq^{-c_2}q^{c_1})}{g_{11}(z/wq^{c_2}q^{c_1})}\\
&\quad= \frac{g_{11}(z/wq^{-c_1-c_2})}{g_{11}(z/wq^{c_1+c_2})}.\\
&\Delta\varphi_1(z)\Delta x_1^+(w)\Delta\varphi_1(z)^{-1}\\
&\quad= \varphi_1(zq^{-c_2/2})\otimes\varphi_1(zq^{c_1/2})(x_1^+(w)\otimes 1 + \varphi_1(wq^{c_1/2})\otimes x_1^+(wq^{c_1}))\\
&\qquad(\varphi_1(zq^{-c_2/2})\otimes\varphi_1(zq^{c_1/2}))^{-1}\\
&\quad= g_{11}(z/wq^{-(c_1+c_2)/2})(x_1^+(w)\otimes 1 + \varphi_1(wq^{c_1/2})\otimes x_1^+(wq^{c_1})).\\
&\Delta\varphi_1(z)\Delta x_1^-(w)\Delta\varphi_1(z)^{-1}\\
&\quad= \varphi_1(zq^{-c_2/2})\otimes\varphi_1(zq^{c_1/2})(1\otimes x_1^-(w) + x_1^-(wq^{c_2})\otimes\psi_1(wq^{c_2/2}))\\
&\qquad\times(\varphi_1(zq^{-c_2/2})\otimes\varphi_1(zq^{c_1/2}))^{-1}\\
&\quad= 1\otimes x_1^-(w)g_{11}(z/wq^{-(c_1+c_2)/2})^{-1}\\
&\qquad+ x_1^-(wq^{c_2})\otimes\psi_1(wq^{c_2/2})g_{11}(z/wq^{(c_1-3c_2)/2})^{-1}\frac{g_{11}(z/wq^{(c_1-3c_2)/2})}{g_{11}(z/wq^{(c_1+c_2)/2})}\\
&\quad= \Delta x_1^-(w)g_{11}(z/wq^{(c_1+c_2)/2})^{-1}.
\end{aligned}$$

The relation between $\Delta\psi_1(z)$ and $\Delta x_1^{\pm}(w)$ can be proved in the same way demonstrated,

$$\begin{aligned}
&[\Delta x_1^+(z), \Delta x_1^-(w)]\\
&\quad= [x_1^+(z)\otimes 1 + \varphi_1(zq^{c_1/2})\otimes x_1^+(zq^{c_1}), 1\otimes x_1^-(w)\\
&\qquad+ x_1^-(wq^{c_2})\otimes\psi_1(wq^{c_2/2})]\\
&\quad= [x_1^+(z)\otimes 1, x_1^-(wq^{c_2})\otimes\psi_1(wq^{c_2/2})]
\end{aligned}$$

$$
\begin{aligned}
&+ [\varphi_1(zq^{c_1/2}) \otimes x_1^+(zq^{c_1}), 1 \otimes x_1^-(w)] \\
&+ [\varphi_1(zq^{c_1/2}) \otimes x_1^+(zq^{c_1}), x_1^-(wq^{c_2}) \otimes \psi_1(wq^{c_2/2})].
\end{aligned}
$$

It is easy to show that the last term above is 0, therefore we have that

$$
\begin{aligned}
&[\Delta x_1^+(z), \Delta x_1^-(w)] \\
&\quad = \frac{(\delta(z/wq^{-c_1-c_2})\psi_1(wq^{c_1/2+c_2}) - \delta(z/wq^{c_1-c_2})\varphi_1(zq^{c_1/2})) \otimes \psi_1(wq^{c_2/2})}{q-q^{-1}} \\
&\qquad + \frac{\varphi_1(zq^{c_1/2}) \otimes (\delta(z/wq^{c_1-c_2})\psi_1(wq^{c_2/2}) - \delta(z/wq^{c_1+c_2})\varphi_1(zq^{c_2/2}))}{q-q^{-1}} \\
&\quad = \frac{\delta(z/wq^{-c_1-c_2})\psi_1(zq^{c_2/2+(c_1+c_2)/2}) \otimes \psi_1(zq^{-c_1/2+(c_1+c_2)/2})}{q-q^{-1}} \\
&\qquad - \frac{\delta(z/wq^{c_1+c_2})\varphi_1(zq^{-c_2/2+(c_1+c_2)/2}) \otimes \varphi_1(zq^{c_1/2+(c_1+c_2)/2})}{q-q^{-1}}
\end{aligned}
$$

$$
\begin{aligned}
&(z-q^2w)\Delta x_1^+(z)\Delta x_1^+(w) \\
&\quad = (z-q^2w)(x_1^+(z) \otimes 1 + \varphi_1(zq^{c_1/2}) \otimes x_1^+(zq^{c_1})) \\
&\qquad \times (x_1^+(w) \otimes 1 + \varphi_1(wq^{c_1/2}) \otimes x_1^+(wq^{c_1})) \\
&\quad = (q^2z-w)x_1^+(w)x_1^+(z) \otimes 1 \\
&\qquad + (z-q^2w)\varphi_1(wq^{c_1/2})x_1^+(z) \otimes x_1^+(wq^{c_1}) \left(\frac{q^2z/w-1}{z/w-q^2}\right) \\
&\qquad + (z-q^2w)x_1^+(w)\varphi_1(zq^{c_1/2}) \otimes x_1^+(zq^{c_1}) \left(\frac{q^2z/w-1}{z/w-q^2}\right) \\
&\qquad + (q^2z-w)\varphi_1(wq^{c_1/2})\varphi_1(zq^{c_1/2}) \otimes x_1^+(wq^{c_1})x_1^+(zq^{c_1}) \\
&\quad = (q^2z-w)\Delta x_1^+(w)\Delta x_1^+(z).
\end{aligned}
$$

Similarly, we can prove the relation between $\Delta x^-(z)$ and $\Delta x^-(w)$.

Let M be the operator from $U_q(\hat{\mathfrak{sl}}_n) \otimes U_q(\hat{\mathfrak{sl}}_n)$ to $U_q(\hat{\mathfrak{sl}}_n)$ defined by the algebra multiplication. We can check that $M(1 \otimes \varepsilon)\Delta = \mathrm{id}$.

$$
\begin{aligned}
M(1 \otimes a)\Delta(x_i^+(z)) &= M(1 \otimes a)(x_i^+(z) \otimes 1 + \varphi_i(zq^{c_1/2}) \otimes x_i^+(zq^{c_1})) \\
&= x_i^+(z) - \varphi_i(zq^{c/2})(\varphi_i(zq^{c/2}))^{-1}x_i^+(z) \\
&= 0 = \varepsilon(x_i^+(z)).
\end{aligned}
$$

Similarly, we can check all the other relations to show that $M(a \otimes 1) = \varepsilon$. Thus, we proved that the comultiplication, the counit and the antipode give a Hopf algebra structure for $U_q(\hat{\mathfrak{sl}}_2)$. □

Presently, there still exist a number of open problems related to this new Hopf algebra structure. One is whether this Hopf algebra is isomorphic, as a Hopf algebra, to the conventional $U_q(\hat{\mathfrak{sl}}_n)$.

3. Bosonization of vertex operators

Vector representation. Let $V = \oplus_{i=0}^{n-1}\mathbb{C}|i\rangle$ be an n-dimensional space, where $\{|i\rangle\}$ is its standard basis. Let $V_z = V \otimes \mathbb{C}[z, z^{-1}]$, where z is a formal variable.

Lemma 3.1. *There exists an n-dimensional representation of $U_q(\hat{\mathfrak{sl}}_n)$ on V_z. The action of the current operators is given by the following:*

(1) $$x_i^+(w)\cdot|j\rangle = \delta_{ij}\delta\left(\frac{w}{q^i z}\right)|i-1\rangle,$$

(2) $$x_i^-(w)\cdot|j\rangle = \delta_{i-1j}\delta\left(\frac{w}{q^i z}\right)|i\rangle,$$

(3) $$\begin{cases} \varphi_i(w)\cdot|i-1\rangle = \dfrac{q^{-1}-q^{(-i+1)}w/z}{1-q^{-i}w/z}|i-1\rangle, \\ \varphi_i(w)\cdot|i\rangle = \dfrac{q-q^{(-i-1)}w/z}{1-q^{-i}w/z}|i\rangle, \qquad \in \mathbb{C}[[w/z]], \\ \varphi_i(w)\cdot|j\rangle = |j\rangle \quad (j \neq i, i-1), \end{cases}$$

(4) $$\begin{cases} \psi_i(w)\cdot|i-1\rangle = \dfrac{q-q^{(i-1)}z/w}{1-q^{i}z/w}|i-1\rangle, \\ \psi_i(w)\cdot|i\rangle = \dfrac{q^{-1}-q^{(i+1)}z/w}{1-q^{i}z/w}|i\rangle, \qquad \in \mathbb{C}[[z/w]], \\ \psi_i(w)\cdot|j\rangle = |j\rangle \quad (j \neq i, i-1), \end{cases}$$

where we set $|-1\rangle = |n\rangle = 0$.

Let $E_{ij}(i, j = 0, 1)$ be the matrix elements such that $E_{ij}|k\rangle = \delta_{kj}|i\rangle$, let $R(z/w)$ be an operator on $V_z \otimes V_w$, which is defined as

$$E_{00}\otimes E_{00} + E_{11}\otimes E_{11} + E_{00}\otimes E_{11}\frac{q-q^{-1}z/w}{1-z/w} + E_{11}\otimes E_{00}\frac{1-z/w}{q^{-1}-q\,z/w}.$$

Proposition 3.2. *Let x be an operator in $U_q(\hat{\mathfrak{sl}}_2)$, then on $V_z \otimes V_w$, we have that $R(z/w)\Delta(x) = \Delta'(x)R(z/w)$, where Δ' is the opposite comultiplication.*

Similarly, we can write the operator $R(z/w)$ for $U_q(\hat{\mathfrak{sl}}_n)$, which is diagonal.

Next, we will give the Frenkel–Jing construction of level 1 representation of $U_q(\hat{\mathfrak{sl}}_n)$ on the Fock space.

Let $\overline{Q} = \oplus_{j=1}^{n-1}\mathbb{Z}\alpha_j$ be the root lattice of $\mathfrak{sl}_n$, $\overline{\Lambda}_j = \Lambda_j - \Lambda_0$ be the classical part of the ith fundamental weight.

Let Heisenberg algebra be an algebra generated by $\{a_{i,k} \mid 1 \leq i \leq n-1, k \in \mathbb{Z}\backslash\{0\}\}$ satisfying

$$[a_{i,k}, a_{j,l}] = \delta_{k+l,0}\frac{[(\alpha_i, \alpha_j)k][k]}{k}.$$

Now let us define a group algebra $\mathbb{C}(q)[\overline{\mathcal{P}}]$. Let $\overline{\mathcal{P}}$ be the weight lattice of $\hat{\mathfrak{sl}}_n$. We fix our free basis $\alpha_2, \dots, \alpha_{n-1}, \overline{\Lambda}_{n-1}$. They satisfy

$$\begin{aligned} \mathrm{e}^{\alpha_i}\mathrm{e}^{\alpha_j} &= (-1)^{(\alpha_i,\alpha_j)}\mathrm{e}^{\alpha_j}\mathrm{e}^{\alpha_i}, \quad 2 \le i, j \le n-1, \\ \mathrm{e}^{\alpha_i}\mathrm{e}^{\overline{\Lambda}_{n-1}} &= (-1)^{\delta_{i,n-1}}\mathrm{e}^{\overline{\Lambda}_{n-1}}\mathrm{e}^{\alpha_i}, \quad 2 \le i \le n-1. \end{aligned}$$

For $\alpha = m_2\alpha_2 + \cdots + m_{n-1}\alpha_{n-1} + m_n\overline{\Lambda}_{n-1}$, we set

$$\mathrm{e}^{\alpha} = \mathrm{e}^{m_2\alpha_2} \dots \mathrm{e}^{m_{n-1}\alpha_{n-1}}\mathrm{e}^{m_n\overline{\Lambda}_{n-1}}.$$

Note that the following equations hold:

$$\begin{aligned} \overline{\Lambda}_i &= -\alpha_{i+1} - 2\alpha_{i+2} - \cdots - (n-i-1)\alpha_{n-1} + (n-i)\overline{\Lambda}_{n-1}, \\ \alpha_1 &= -2\alpha_2 - 3\alpha_3 - \cdots - (n-1)\alpha_{n-1} + n\overline{\Lambda}_{n-1}. \end{aligned}$$

Put

$$a^*_{ik} = \frac{1}{[k]^2[nk]}\sum_{j=1}^{n-1}[\min(i,j)k][\min(n-i,n-j)k]a_{jk}, \quad 1 \le i < n, \quad k \neq 0.$$

Then they satisfy

$$[a^*_{ik}, a_{jl}] = \delta_{ij}\delta_{k+l,0}\frac{[k]}{k}.$$

We define the Fock space as

$$\mathcal{F}_i := \mathbb{C}(q)[a_{j,-k}(1 \le j \le n-1, k \in \mathbb{Z}_{>0})] \otimes \mathbb{C}(q)[\overline{Q}]\mathrm{e}^{\overline{\Lambda}_i} \quad (0 \le i \le n-1).$$

The action of operators $a_{j,k}, \partial_{\alpha_j}, \mathrm{e}^{\alpha}(1 \le j \le n-1, \alpha \in \overline{Q})$ is given by

$$\begin{aligned} a_{j,k} \cdot f \otimes \mathrm{e}^{\beta} &= \begin{cases} a_{j,k} f \otimes \mathrm{e}^{\beta}, & k < 0, \\ [a_{j,k}, f] \otimes \mathrm{e}^{\beta}, & k > 0, \end{cases} \\ \partial_{\alpha} \cdot f \otimes \mathrm{e}^{\beta} &= (\alpha, \beta) f \otimes \mathrm{e}^{\beta} \quad \text{for } f \otimes \mathrm{e}^{\beta} \in \mathcal{F}_i, \\ \mathrm{e}^{\alpha} \cdot f \otimes \mathrm{e}^{\beta} &= f \otimes \mathrm{e}^{\alpha}\mathrm{e}^{\beta}. \end{aligned}$$

Lemma 3.3. *Let*

$$\begin{aligned} x^{\pm}_j(z) &\mapsto \exp\left[\pm\sum_{k>0}\frac{a_{j,-k}}{[k]}q^{\mp k/2}z^k\right]\exp\left[\mp\sum_{k>0}\frac{a_{j,k}}{[k]}q^{\mp k/2}z^{-k}\right]\mathrm{e}^{\pm\alpha_j}z^{\pm\partial_{\alpha_j}+1}, \\ \varphi_j(z) &\mapsto \exp\left[-(q-q^{-1})\sum_{k>0}a_{j,-k}z^k\right]q^{-\partial_{\alpha_j}}, \\ \psi_j(z) &\mapsto \exp\left[(q-q^{-1})\sum_{k>0}a_{j,k}z^{-k}\right]q^{\partial_{\alpha_j}}. \end{aligned}$$

These give level 1 highest representations with the ith fundamental weight on $\mathcal{F}_i$.

Definition 3.4. Vertex operators are intertwiners of the following types:
(i) Type I $\quad \Phi^{(i,i+1)}(z) : \mathcal{F}_{i+1} \mapsto \mathcal{F}_i \otimes V_z$,
(ii) Type II $\quad \Psi^{(i,i+1)}(z) : \mathcal{F}_{i+1} \mapsto V_z \otimes \mathcal{F}_i$,
(iii) Dual of type I $\quad \Phi^{*(i+1,i)}(z) : \mathcal{F}_i \otimes V_z \mapsto \mathcal{F}_{i+1}$,
(iv) Dual of type II $\quad \Psi^{*(i+1,i)}(z) : V_z \otimes \mathcal{F}_z \mapsto \mathcal{F}_{i+1}$.
Here the indices are considered modulo n.

Set

$$\Phi^{(i,i+1)}(z) = \sum_{j=0}^{n-1} \Phi_j^{(i,i+1)}(z) \otimes |j\rangle, \qquad \Psi^{(i,i+1)}(z) = \sum_{j=0}^{n-1} |j\rangle \otimes \Psi_j^{(i,i+1)}(z),$$

$$\Phi^{*(i+1,i)}(z)(u \otimes |j\rangle) = \Phi_j^{*(i+1,i)}(z)u,$$

$$\Psi^{*(i+1,i)}(z)(|j\rangle \otimes u) = \Psi_j^{*(i+1,i)}(z)u, \quad u \in \mathcal{F}_{i+1}.$$

The normalization of these operators is given by
(i) $\langle \Lambda_i | \Phi_i^{(i,i+1)}(z) | \Lambda_{i+1} \rangle = 1, \qquad \langle \Lambda_i | \Psi_i^{(i,i+1)}(z) | \Lambda_{i+1} \rangle = 1,$
(ii) $\langle \Lambda_{i+1} | \Phi_i^{*(i+1,i)}(z) | \Lambda_i \rangle = 1, \qquad \langle \Lambda_{i+1} | \Psi_i^{*(i+1,i)}(z) | \Lambda_i \rangle = 1.$

We will first give the list of OPEs (operator product expansions), where we abbreviate the superscript of vertex operators.

Type I, II

$$[\Phi_j(z), x_i^+(w)] = \delta_{i-1,j}\delta\left(\frac{w}{q^{i-1}z}\right)\varphi_i(wq^{1/2})\Phi_i(z),$$

$$\Phi_j(z)x_i^-(w) = \begin{cases} \dfrac{q - q^{(i-1)}z/w}{1 - q^i\, z/w} x_i^-(w)\Phi_{i-1}(z) & \text{for } j = i-1, \\ \dfrac{q^{-1} - q^{(i+1)}z/w}{1 - q^i\, z/w} x_i^-(w)\Phi_i(z) & \\ \quad + \delta\left(\dfrac{w}{q^i z}\right)\Phi_{i-1}(z) & \text{for } j = i, \\ x_i^-(w)\Phi_j(z) & \text{otherwise.} \end{cases}$$

$$\Phi_j(z)\varphi_i(w) = \begin{cases} \dfrac{q^{-1} - q^{(-i+3/2)}w/z}{1 - q^{(-i+1/2)}w/z}\varphi_i(w)\Phi_{i-1}(z) & \text{for } j = i-1, \\ \dfrac{q - q^{(-i-1/2)}w/z}{1 - q^{(-i+1/2)}w/z}\varphi_i(w)\Phi_i(z) & \text{for } j = i, \\ \varphi_i(w)\Phi_j(z) & \text{otherwise.} \end{cases}$$

$$\Phi_j(z)\psi_i(w) = \begin{cases} \dfrac{q - q^{(i-1/2)}z/w}{1 - q^{(i+1/2)}z/w}\psi_i(w)\Phi_{i-1}(z) & \text{for } j = i-1, \\ \dfrac{q^{-1} - q^{(i+3/2)}z/w}{1 - q^{(i+1/2)}z/w}\psi_i(w)\Phi_i(z) & \text{for } j = i, \\ \psi_i(w)\Phi_j(z) & \text{otherwise.} \end{cases}$$

$$\Psi_j(z)x_i^+(w) = \begin{cases} \dfrac{q^{-1} - q^{(-i+1)}w/z}{1 - q^{-i}w/z}x_i^+(w)\Psi_{i-1}(z) & \text{for } j = i-1, \\ \quad +\delta\left(\dfrac{w}{q^i z}\right)\Psi_i(z) & \\ \dfrac{q - q^{(-i-1)}w/z}{1 - q^{-i}w/z}x_i^+(w)\Psi_i(z) & \text{for } j = i, \\ x_i^+(w)\Psi_j(z) & \text{otherwise.} \end{cases}$$

$$[\Psi_j(z), x_i^-(w)] = \delta_{i,j}\delta\left(\frac{w}{q^{i-1}z}\right)\psi_i(wq^{1/2})\Psi_{i-1}(z),$$

$$\Psi_j(z)\varphi_i(w) = \begin{cases} \dfrac{q^{-1} - q^{(-i+1/2)}w/z}{1 - q^{(-i-1/2)}w/z}\varphi_i(w)\Psi_{i-1}(z) & \text{for } j = i-1, \\ \dfrac{q - q^{(-i-3/2)}w/z}{1 - q^{(-i-1/2)}w/z}\varphi_i(w)\Psi_i(z) & \text{for } j = i, \\ \varphi_i(w)\Psi_j(z) & \text{otherwise.} \end{cases}$$

$$\Psi_j(z)\psi_i(w) = \begin{cases} \dfrac{q - q^{(i-3/2)}z/w}{1 - q^{(i-1/2)}z/w}\psi_i(w)\Psi_{i-1}(z) & \text{for } j = i-1, \\ \dfrac{q^{-1} - q^{(i+1/2)}z/w}{1 - q^{(i-1/2)}z/w}\psi_i(w)\Psi_i(z) & \text{for } j = i, \\ \psi_i(w)\Psi_j(z) & \text{otherwise.} \end{cases}$$

Dual of type I, II

$$[x_i^+(w), \Phi_j^*(z)] = \delta_{i,j}\delta\left(\frac{w}{q^{i-1}z}\right)\Phi_{i-1}^*(z)\varphi_i(wq^{1/2}),$$

$$x_i^-(w)\Phi_j^*(z) = \begin{cases} \dfrac{q - q^{(i-1)}z/w}{1 - q^i z/w}\Phi_{i-1}^*(z)x_i^-(w) & \text{for } j = i-1, \\ \quad +\delta\left(\dfrac{w}{q^i z}\right)\Phi_i^*(z) & \\ \dfrac{q^{-1} - q^{(i+1)}z/w}{1 - q^i z/w}\Phi_i^*(z)x_i^-(w) & \text{for } j = i, \\ \Phi_j^*(z)x_i^-(w) & \text{otherwise.} \end{cases}$$

$$\varphi_i(w)\Phi_j^*(z) = \begin{cases} \dfrac{q^{-1}-q^{(-i+3/2)}w/z}{1-q^{(-i+1/2)}w/z}\Phi_{i-1}^*(z)\varphi_i(w) & \text{for } j=i-1, \\ \dfrac{q-q^{(-i-1/2)}w/z}{1-q^{(-i+1/2)}w/z}\Phi_i^*(z)\varphi_i(w) & \text{for } j=i, \\ \Phi_j^*(z)\varphi_i(w) & \text{otherwise.} \end{cases}$$

$$\psi_i(w)\Phi_j^*(z) = \begin{cases} \dfrac{q-q^{(i-1/2)}z/w}{1-q^{(i+1/2)}z/w}\Phi_{i-1}^*(z)\psi_i(w) & \text{for } j=i-1, \\ \dfrac{q^{-1}-q^{(i+3/2)}z/w}{1-q^{(i+1/2)}z/w}\Phi_i^*(z)\psi_i(w) & \text{for } j=i, \\ \Phi_j^*(z)\psi_i(w) & \text{otherwise.} \end{cases}$$

$$x_i^+(w)\Psi_j^*(z) = \begin{cases} \dfrac{q^{-1}-q^{(-i+1)}w/z}{1-q^{-i}\,w/z}\Psi_{i-1}^*(z)x_i^+(w) & \text{for } j=i-1, \\ \dfrac{q-q^{(-i-1)}w/z}{1-q^{-i}\,w/z}\Psi_i^*(z)x_i^+(w) & \text{for } j=i, \\ \quad +\delta\left(\dfrac{w}{q^i z}\right)\Psi_{i-1}^*(z) & \\ \Psi_j^*(z)x_i^+(w) & \text{otherwise.} \end{cases}$$

$$[x_i^-(w), \Psi_j^*(z)] = \delta_{i-1,j}\delta\left(\frac{w}{q^{i-1}z}\right)\Psi_i^*(z)\psi_i(wq^{1/2}),$$

$$\varphi_i(w)\psi_j^*(z) = \begin{cases} \dfrac{q^{-1}-q^{(-i+1/2)}w/z}{1-q^{(-i-1/2)}w/z}\Psi_{i-1}^*(z)\varphi_i(w) & \text{for } j=i-1, \\ \dfrac{q-q^{(-i-3/2)}w/z}{1-q^{(-i-1/2)}w/z}\Psi_i^*(z)\varphi_i(w) & \text{for } j=i, \\ \Psi_j^*(z)\varphi_i(w) & \text{otherwise.} \end{cases}$$

$$\psi_i(w)\Psi_j^*(z) = \begin{cases} \dfrac{q-q^{(i-3/2)}z/w}{1-q^{(i-1/2)}z/w}\Psi_{i-1}^*(z)\psi_i(w) & \text{for } j=i-1, \\ \dfrac{q^{-1}-q^{(i+1/2)}z/w}{1-q^{(i-1/2)}z/w}\Psi_i^*(z)\psi_i(w) & \text{for } j=i, \\ \Psi_j^*(z)\psi_i(w) & \text{otherwise.} \end{cases}$$

Remark. We remark that two OPEs related to $\Psi_i(z)x_i^-(w)$ and $x_i^+(w)\Phi_i^*(z)$ are

$$\Psi_i(z)x_i^-(w) = x_i^-(w)\Psi_i(z), \qquad x_i^+(w)\Phi_i^*(z) = \Phi_i^*(z)x_i^+(w).$$

However, the two sides of the equations are in different expansion directions, which implies that both sides have poles. These two equations will not degenerate the classical intertwiner relations when $q = 1$. For this case, we impose the following locality conditions:

$$\left(1-\frac{w}{q^{i+1}z}\right)[\Psi_i(z), x_i^-(w)] = 0, \qquad \left(1-\frac{w}{q^{i+1}z}\right)[x_i^+(w), \Phi_i^*(z)] = 0,$$

which ensure that the degenertion of these formulae is consistent with its classical degeneration when $q = 1$.

Theorem 3.5. *The vertex operators are given as:*

$$\begin{aligned}
\Phi_j^{(i,i+1)}(z) &= \exp\left[-\sum_{k>0} a^*_{j-k} q^{3k/2}(q^j z)^k + \sum_{k>0} a^*_{j+1-k} q^{k/2}(q^j z)^k\right] \\
&\quad \times \exp\left[-\sum_{k>0} a^*_{jk} q^{-k/2}(q^j z)^{-k} + \sum_{k>0} a^*_{j+1k} q^{k/2}(q^j z)^{-k}\right] \\
&\quad \times e^{\overline{\Lambda}_j - \overline{\Lambda}_{j+1}} (q^{j+1}z)^{\partial_{\overline{\Lambda}_j}} (q^j z)^{-\partial_{\overline{\Lambda}_{j+1}}} (-1)^{(n-1)\partial_{\overline{\Lambda}_1}} (c)^i_j, \\
\Psi_j^{(i,i+1)}(z) &= \exp\left[-\sum_{k>0} a^*_{j-k} q^{k/2}(q^j z)^k + \sum_{k>0} a^*_{j+1-k} q^{-k/2}(q^j z)^k\right] \\
&\quad \times \exp\left[-\sum_{k>0} a^*_{jk} q^{-3/2k}(q^j z)^{-k} + \sum_{k>0} a^*_{j+1k} q^{-k/2}(q^j z)^{-k}\right] \\
&\quad \times e^{\overline{\Lambda}_j - \overline{\Lambda}_{j+1}} (q^{j+1}z)^{\partial_{\overline{\Lambda}_j}} (q^j z)^{-\partial_{\overline{\Lambda}_{j+1}}} (-1)^{(n-1)\partial_{\overline{\Lambda}_1}} (c)^i_j, \\
\Phi_j^{*(i+1,i)}(z) &= \exp\left[\sum_{k>0} a^*_{j-k} q^{3/2k}(q^j z)^k - \sum_{k>0} a^*_{j+1-k} q^{k/2}(q^j z)^k\right] \\
&\quad \times \exp\left[\sum_{k>0} a^*_{jk} q^{-k/2}(q^j z)^{-k} - \sum_{k>0} a^*_{j+1k} q^{k/2}(q^j z)^{-k}\right] \\
&\quad \times e^{-\overline{\Lambda}_j + \overline{\Lambda}_{j+1}} (q^{j+1}z)^{-\partial_{\overline{\Lambda}_j}} (q^j z)^{\partial_{\overline{\Lambda}_{j+1}}} (-1)^{-(n-1)\partial_{\overline{\Lambda}_1}} (c^*)^i_j, \\
\Psi_j^{*(i+1,i)}(z) &= \exp\left[\sum_{k>0} a^*_{j-k} q^{k/2}(q^j z)^k - \sum_{k>0} a^*_{j+1-k} q^{-k/2}(q^j z)^k\right] \\
&\quad \times \exp\left[\sum_{k>0} a^*_{jk} q^{-3k/2}(q^j z)^{-k} - \sum_{k>0} a^*_{j+1k} q^{-k/2}(q^j z)^{-k}\right] \\
&\quad \times e^{-\overline{\Lambda}_j + \overline{\Lambda}_{j+1}} (q^{j+1}z)^{-\partial_{\overline{\Lambda}_j}} (q^j z)^{\partial_{\overline{\Lambda}_{j+1}}} (-1)^{-(n-1)\partial_{\overline{\Lambda}_1}} (c^*)^i_j,
\end{aligned}$$

where $(c)^i_j = (-q)^{j-i}(c)^i_i$, $(c^*)^i_j = (c^*)^i_i$ *and*

$$\begin{aligned}
(c)^i_i &= [(-1)^{-(n-1)}z]^{(n-i-1)/n}(-1)^{(n-i-1)(n-i-2)/2}, \\
(c^*)^i_i &= [(-1)^{n-1}]^{(n-i)/n}[q^n z]^{i/n}(-1)^{(n-i)(n-i-1)/2}.
\end{aligned}$$

Here $*$ *means dual.*

From this theorem, we can derive all the correlation functions of vertex operators. We will give a simple example of the case of $U_q(\hat{\mathfrak{sl}}_2)$ to show what they are like.

Let υ be the highest weigh vector in $\mathcal{F}_0$:

$$\langle \upsilon, \Phi_0^{(0,1)}(z_1)\Phi_0^{(1,0)}(z_2), \upsilon\rangle = \langle \upsilon, \Phi_1^{(0,1)}(z_1)\Phi_1^{(1,0)}(z_2), \upsilon\rangle = 0,$$

$$\langle \upsilon, \Phi_1^{(0,1)}(z_2)\Phi_0^{(1,0)}(z_1), \upsilon\rangle = -q^{-1}(z_1/z_2)^{1/2}\frac{(qz_1/z_2; q^4)_\infty}{(q^3z_1/z_2, q^4)_\infty},$$

$$\langle \upsilon, \Phi_0^{(0,1)}(z_2)\Phi_1^{(1,0)}(z_1), \upsilon\rangle = z_1^{1/2}z_2^{3/2}\frac{(q^4z_1/z_2; q^4)_\infty}{(q^6z_1/z_2, q^4)_\infty},$$

where $(z, p)_\infty = \prod_0^\infty(1 - zp^m)$.

With these vertex operators, we can derive the following formulae.

Theorem 3.6.

(1) $:\Phi_{i-1}^{(i-1,i)}(z)\Phi_i^{*(i+1,i)}(z) := x_i^-(q^iz),$

(2) $:\Psi_i^{(i-1,i)}(z)\Psi_{i-1}^{*(i+1,i)}(z) := -qx_i^+(q^iz),$

(3) $:x_i^+(q^{1/2}z)x_i^-(q^{-1/2}z) := z^2\psi_i(z),$

(4) $:x_i^+(q^{-1/2}z)x_i^-(q^{1/2}z) := z^2\varphi_i(z),$

With $:\cdot:$ *is the normal ordering.*

In [Di1,Di2], we showed that the vertex operators in the classical case have the structure of Clifford algebras, which is used to construct representations of $\hat{\mathfrak{sl}}_n$. This is the boson-fermion correspondence. We also established a quantum version of boson–fermion correspondence, where the quantum fermions are identified as vertex operators with the conventional comultiplication. However, the quantum boson–fermion correspondence is very implicit due to the incomplete formulae for intertwiners. With Theorems 3.5 and 3.6, the correlation functions of all the vertex operators are clear. Utilizing all these results, we hope that we can present a different definition of quantum Clifford algebras and derive a more explicit boson–fermion correspondence. Furthermore we recognize that our formulae may be related to the quantum $\mathcal{W}$-algebras [AKOS,FF] especially the bosonization of these algebras.

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